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HIDDEN Z-MATRICES WITH POSITIVE PRINCIPAL MINORS.(U)

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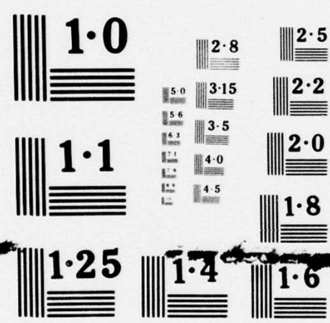

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HIDDEN Z-MATRICES WITH POSITIVE PRINCIPAL MINORS

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UNIVERSITY OF WISCONSIN - MADISON
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HIDDEN Z-MATRICES WITH POSITIVE PRINCIPAL MINORS

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ABSTRACT

Let $\mathcal{C}$ denote the class of hidden Z-matrices, i.e., $M \in \mathcal{C}$ if and only if there exist Z-matrices X and Y such that the following two conditions are satisfied:

(M1) $MX = Y$

(M2) $r^T X + s^T Y > 0$ for some $r, s \geq 0$.

Let P denote the class of real square matrices having positive principal minors. The class $\mathcal{C}$ arises recently as a generalization of the class of Z-matrices [9], [23], [24]. In this paper, we explore various matrix-theoretic aspects of the class $\mathcal{C} \cap P$.

AMS(MOS) Subject Classification - 1502, 15A24, 15A45, 90C99

Key Words - Hidden Leontief, Hidden Z, Linear complementarity, Positivity of principal minors,

Work Unit Number 2 - Matrix Theory

EXPLANATION

Matrix theory has been playing a very important role in the theory and applications of the linear complementarity problem. It is especially useful in the development of efficient algorithms for solving large-scale linear complementarity problems. Part of its usefulness is due to the fact that it allows the matrix structures which might be inherent in the problems, to be exploited profitably. Among the classes of

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matrices which arise from the various applications of the linear complementarity problem is the class of K-matrices, i.e. real square matrices whose off-diagonal entries are nonpositive (so-called Z-matrices) and whose principal minors are all positive (so-called P-matrices). The class of K-matrices also plays very important roles in other fields.

Recently, an extension of the class of Z-matrices, denoted by $\mathcal{C}$, was introduced by Mangasarian who showed that a solution to a linear complementarity problem with a matrix in $\mathcal{C}$ can be obtained, numerically, by solving a suitable linear program. Mangasarian's result has later been refined by R. W. Cottle and the author. The purpose of this paper is to explore various matrix-theoretic aspects of matrices in $\mathcal{C}$ which are P-matrices as well.

HIDDEN Z-MATRICES WITH POSITIVE PRINCIPAL MINORS

Jong-Shi Pang

1. INTRODUCTION. The theory and applications of the linear complementarity problem with a Z-matrix (i.e. a real square matrix whose off-diagonal entries are non-positive) have received much attention in the literature [1], [16], [20], [27], [28], [31], [37], [41]. A subclass of the class of Z-matrices, which is particularly important in the linear complementarity problem (and also in many other areas) is the class of K-matrices i.e. Z-matrices that are also P-matrices (real square matrices having positive principal minors). The theory and applications of the linear complementarity problem with a K-matrix have been documented in many places in the literature [3], [5], [7], [8], [11], [12], [36]. A major difference between a linear complementarity problem with a Z-matrix and that with a K-matrix is that the former problem is not always feasible; whereas the latter problem always has a unique solution [12], [39].

Recently, an extension of the class of Z-matrices was introduced by Mangasarian in his study of solving linear complementarity problems as linear programs. See [23], [24]. This is the class $\mathbf{C}$ of real square matrices M for which there exist Z-matrices X and Y satisfying the following two conditions:

(M1) $MX = Y$

(M2) $r^T X + s^T Y > 0$ for some $r, s \geq 0$.

Mangasarian's results in [23], [24] have been refined and extended by R. W. Cottle and the author [9], [10], by Mangasarian himself [25] and by the author [33]. Some basic properties of matrices belonging to $\mathbf{C}$ have been obtained in [9]. It is clear that if M is a Z-matrix, then $M \in \mathbf{C}$. Several other classes of matrices belonging to $\mathbf{C}$ are given in [9], [25], [31], [32]. The class $\mathbf{C}$ appears to be a very appropriate generalization of the class Z^*, because for one thing, many of the properties originally possessed by a Z-matrix are carried over to matrices in $\mathbf{C}$. This is particularly true in the contexts of the linear complementarity problem and of the Leontief substitution systems. See [34]. We propose to call matrices in $\mathbf{C}$ hidden Z-matrices. The word "hidden" is borrowed from

* The letters K, P and Z will also denote the corresponding classes of matrices.

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that in "hidden Leontief matrices." These hidden Leontief matrices were introduced by Saigal [38] in his study of a generalized Leontief property of rectangular matrices. Recall that an $n \times m$ matrix A is said to be Leontief ([14], [43]) if it has at most one positive entry in each column and there is a vector $x > 0$ such that $Ax > 0$. It is clear that if M is a Z-matrix, then the matrix (I, M^T) is Leontief. Slightly modifying the definitions in [22], [38], we say that an $n \times m$-matrix A is hidden Leontief if there exists an $n \times n$ nonsingular matrix D such that DA is Leontief. It has been shown [34, Prop. 4.1] that if the matrix M is hidden Z, then the matrix (I, M^T) is hidden Leontief. The matrix (I, M^T) arises naturally in the linear programming formulation of a linear complementarity problem with a hidden Z-matrix M. See [34].

Numerous equivalent conditions under which a Z-matrix will become a K-matrix have been surveyed in [17]. It is very natural to ask the question: What are some of the matrix-theoretic properties of the class of K-matrices that are carried over to the class $\mathbf{C} \cap P$? Therefore the purpose of this paper is to provide at least a partial answer to this question by exploring various matrix-theoretic aspects of matrices belonging to $\mathbf{C} \cap P$. The essential result is a theorem which provides a necessary and sufficient condition for a hidden Z-matrix to be a P-matrix. As an application of this characterization, we shall establish a representation theorem for matrices in $\mathbf{C} \cap P$ and identify several classes of matrices belonging to $\mathbf{C} \cap P$.

2. THE CLASS $\mathbf{C} \cap P$.

We start by explaining the notations and reviewing some facts to be used later. We denote the class of all $n \times m$ real matrices by $R^{n\times m}$. We denote the cardinality of a set S by $|S|$. Let $M \in R^{n\times m}$ and $\alpha, \beta \subseteq \{1,\dots,n\}$. We define

$$M_{\alpha\beta} = \begin{bmatrix} m_{\alpha_1\beta_1} & \cdots & m_{\alpha_1\beta_t} \\ \vdots & & \vdots \\ m_{\alpha_s\beta_1} & \cdots & m_{\alpha_s\beta_t} \end{bmatrix}$$

where $\alpha = \{\alpha_1,\dots,\alpha_s\}$ and $\beta = \{\beta_1,\dots,\beta_t\}$ with $1 \le \alpha_1 < \dots < \alpha_s \le n$ and $1 \le j_1 < \dots < j_t \le n$. In particular, $M_{\alpha\alpha}$ is a principal submatrix of M. Similarly, if q is an n-vector, we define $q_\alpha = (q_{\alpha_1},\dots,q_{\alpha_s})^T$. Let M be a square matrix. By a <u>principal rearrangement</u> of M, we mean a matrix $\bar{M} = P^T MP$ where P is a permutation matrix. Clearly, the classes of K-, P- and Z-matrices are invariant under principal rearrangements. Moreover, the property of a matrix belonging to any one of the three classes K, P and Z is inherited by each of its principal submatrices. Let A be a nonsingular principal submatrix of a square matrix M. Let $\bar{M}$ be a principal rearrangement of M such that $\bar{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then the <u>Schur complement of</u> A <u>in</u> M, denoted by (M/A) is the matrix $D - CA^{-1}B$. Properties and applications of the Schur complements have been surveyed in [4]. It has been proved [13] that if M is a K-matrix, then so is every Schur complement in M. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of a matrix $M \in R^{n\times n}$. Let P be a permutation matrix such that $P^T MP = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}$ where $\beta = \{1,\dots,n\} \setminus \alpha$. The matrix $\hat{M} \equiv PM^*P^T$ where

$$M^* = \begin{pmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\beta} \\ M_{\beta\alpha} M_{\alpha\alpha}^{-1} & M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta} \end{pmatrix}$$

is called a <u>principal pivot transform</u> of M. The matrix M^* is obtained from M by performing a block pivot on $M_{\alpha\alpha}$. Properties and applications of the principal pivot transforms are well recognized in mathematical programming [2], [6], [19], [42]. It has been shown [6] that if M is a P-matrix, then so is each of its principal pivot transforms. Note that every Schur complement in M appears as a principal submatrix of a principal pivot transform of M.

A nonnegative matrix $Q \in R^{n \times n}$ is said to be sub-stochastic if $Qe < e$ where e is the vector of 1's. Clearly, if Q is sub-stochastic, then the matrix $I - Q \in K$. A matrix $M \in R^{n \times n}$ is said to be an S-matrix [18] if there exists a vector $x > 0$ such that $Mx > 0$. It has been shown that every P-matrix is an S-matrix [18] and that every Z-matrix which is also an S-matrix is indeed a K-matrix [17]. Let $A \in R^{n \times m}$ be Leontief. It is said to be totally Leontief if there exists a vector $y \geq 0$ such that $y^T A > 0$. Clearly, if M is a K-matrix, then the matrix (I, M^T) is totally Leontief. Let $A \in R^{n \times m}$. It is said to be hidden totally Leontief if there is an $n \times n$ nonsingular matrix D such that DA is totally Leontief.

A pair of real square matrices (A,B) of the same order n is said to have the $\mathcal{P}$-property if for any two complementary index sets α and β in $\{1,\ldots,n\}$, the matrix $\begin{pmatrix} A_{\alpha\alpha} & B_{\alpha\beta} \\ A_{\beta\alpha} & B_{\beta\beta} \end{pmatrix}$ is in P. In particular, both A and B are in P. The $\mathcal{P}$-property was introduced recently by I. Kaneko [21] in his study of a special class of linear complementarity problems with applications to certain structural engineering problems. It is clear that a matrix M is a P-matrix if and only if the pair of matrices (I, M^T) has the $\mathcal{P}$-property.

We are now ready to establish our results. The first one is the main theorem which provides a necessary and sufficient condition for a hidden Z-matrix to be a P-matrix. The theorem generalizes the fact that a Z-matrix is in P if and only if it is an S-matrix.

Theorem 1. Let $M \in \mathbf{C} \cap R^{n \times n}$. Then the following are equivalent:

(1) M is a P-matrix.

(2) M is an S-matrix.

Proof: (1) $\Rightarrow$ (2). This is true regardless of what M is and has been mentioned above.

(2) $\Rightarrow$ (1). This is the non-trivial part of the theorem and is of fundamental importance throughout the paper. We use induction on n. The implication is obviously true for $n = 1$. Suppose that it is true for all matrices of order $< n$. Consider a matrix $M \in \mathbf{C} \cap R^{n \times n}$ which is an S-matrix as well. Let X and Y be Z-matrices satisfying the defining conditions (M1) and (M2). According to [9, Thm. 3.9], the matrix X is nonsingular and the matrix (X^T, Y^T) is Leontief. Since M is an S-matrix, there exists a vector $v \in R^n$ such that $Xv > 0$ and $Yv > 0$. Since (X^T, Y^T) is Leontief,

such a vector v must be positive (see [14] e.g.). Hence, if α and β are any two complementary index sets in $\{1,\ldots,n\}$, the matrix $\begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ Y_{\beta\alpha} & Y_{\beta\beta} \end{pmatrix}$ is in K. In particular, we have $\det X > 0$ and $\det Y > 0$. Therefore $\det M > 0$. Thus it remains to show that every proper principal submatrix of M has positive determinant. To prove this, it suffices to show that if $M_{\alpha\alpha}$ is a proper principal submatrix of M, then $M_{\alpha\alpha}$ satisfies the assumptions in the induction hypothesis. In other words, we need to show that $M_{\alpha\alpha} \in \mathbf{C} \cap R^{|\alpha|}$ and there exists a vector $y \in R^{|\alpha|}$ such that $y > 0$ and $M_{\alpha\alpha} y > 0$. Since every principal rearrangement of M belongs to $\mathbf{C}$ (with X and Y rearranged accordingly) and is obviously an S-matrix, we may assume, without loss of generality, that $M_{\alpha\alpha}$ is a leading principal submatrix of M. Let $\beta = \{1,\ldots,n\} \setminus \alpha$. We have

$$(M_{\alpha\alpha} \; M_{\alpha\beta}) \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = (Y_{\alpha\alpha} \; Y_{\alpha\beta}) .$$

By an easy calculation, we may deduce

$$M_{\alpha\alpha}(X_{\alpha\alpha} - X_{\alpha\beta} X_{\beta\beta}^{-1} X_{\beta\alpha}) = Y_{\alpha\alpha} - Y_{\alpha\beta} X_{\beta\beta}^{-1} X_{\beta\alpha}$$

or equivalently,

(i) $$M_{\alpha\alpha}(X/X_{\beta\beta}) = (W/X_{\beta\beta})$$

where

$$W = \begin{pmatrix} Y_{\alpha\alpha} & Y_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} .$$

Since $(X/X_{\beta\beta})$ and $(W/X_{\beta\beta})$ are both K-matrices, it follows that $M_{\alpha\alpha} \in \mathbf{C}$. Finally, since X is a K-matrix, we have

$$(X/X_{\beta\beta})v_\alpha = ((X/X_{\beta\beta}) \;\; 0) \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix}$$

$$= (I \;\; -X_{\alpha\beta}X_{\beta\beta}^{-1}) \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} > 0 .$$

Similarly, we obtain

$$(W/X_{\beta\beta})v_\alpha > 0 .$$

Let $y = (X/X_{\beta\beta})v_\alpha$, then

$$M_{\alpha\alpha} y > 0 \text{ and } y > 0 .$$

Therefore, $M_{\alpha\alpha}$ satisfies the assumptions in the induction hypothesis. This completes the inductive step and also the proof of the theorem.

Corollary 1. If $M \in \mathcal{C} \cap P$, then the matrix $A^T = (I, M^T)$ is hidden totally Leontief.

Proof: In fact, if M satisfies conditions (M1) and (M2) for Z-matrices X and Y, then condition (2) is equivalent to the fact that the matrix $(X^T, Y^T) = X^T A^T$ is totally Leontief. The conclusion of the corollary is therefore an immediate consequence of Theorem 1.

If the P-matrix M satisfies conditions (M1) and (M2) for Z-matrices X and Y, the proof of Theorem 1 shows that the pair of matrices (X^T, Y^T) has the ρ-property. The converse is also true and is contained in Proposition 1 below. The proposition generalizes the fact that a P-matrix must necessarily be an S-matrix. The proof of the proposition depends on the lemma below whose proof is omitted but can be found in [21].

Lemma 1. (Kaneko [21]) Let (A,B) have the ρ-property. Then for every q and $a > 0$, there exists a unique solution $\binom{v}{x}$ to the problem:

(iia) $\qquad w = q + Av + Bx \geq 0 \qquad v \geq 0$

(iib) $\qquad z = a - v \geq 0 \qquad x \geq 0$

(iic) $\qquad w^T v = z^T x = 0$

Proposition 1. Let $X, Y \in R^{n \times n}$. If (X^T, Y^T) has the ρ-property, then there exists a vector v such that $Xv > 0$ and $Yv > 0$.

Proof*: Suppose there exist no such vectors v. Then by Gordon's alternative theorem on the feasibility of a homogeneous system of linear equations [15, Thm. 5, p. 136], it follows that there exist nonnegative vectors r and s, not both vanishing such that

$$X^T r + Y^T s = 0 .$$

Let $\alpha = \{i: r_i = 0\}$ and $\beta = \{1,\ldots,n\} \setminus \alpha$. Then we have

* The author is grateful to Professor I. Kaneko and Mr. W. Hallman for some valuable discussion on this proof.

$$(X^T)_{\alpha\beta} r_\beta + (Y^T)_{\alpha\alpha} s_\alpha + (Y^T)_{\alpha\beta} s_\beta = 0$$

which gives

$$s_\alpha = -(Y^T)^{-1}_{\alpha\alpha} (X^T)_{\alpha\beta} r_\beta - (Y^T)^{-1}_{\alpha\alpha} (Y^T)_{\alpha\beta} s_\beta .$$

Therefore, the vector $\binom{r_\beta}{s_\beta}$ is a solution to the problem (ii) with $q = 0$, $a = r_\beta$, $A = (X^T)_{\beta\beta} - (Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} (X^T)_{\alpha\beta}$ and $B = (Y^T/(Y^T)_{\alpha\alpha})$. Obviously, the zero vector is also a solution to the same problem. It is not hard to verify that the pair of matrices

$$\begin{pmatrix} (Y^T)_{\alpha\alpha} & (X^T)_{\alpha\beta} \\ 0 & (X^T)_{\beta\beta} - (Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} (X^T)_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -(Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} & I \end{pmatrix} \begin{pmatrix} (Y^T)_{\alpha\alpha} & (X^T)_{\alpha\beta} \\ (Y^T)_{\beta\alpha} & (X^T)_{\beta\beta} \end{pmatrix}$$

and

$$\begin{pmatrix} (Y^T)_{\alpha\alpha} & (Y^T)_{\alpha\beta} \\ 0 & (Y^T)_{\beta\beta} - (Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} (Y^T)_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -(Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} & I \end{pmatrix} \begin{pmatrix} (Y^T)_{\alpha\alpha} & (Y^T)_{\alpha\beta} \\ (Y^T)_{\beta\alpha} & (Y^T)_{\beta\beta} \end{pmatrix}$$

also has the $\mathcal{P}$-property, therefore so does the pair $((X^T)_{\beta\beta} - (Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} (X^T)_{\alpha\beta}$, $(Y^T)_{\beta\beta} - (Y^T)_{\beta\alpha} (Y^T)^{-1}_{\alpha\alpha} (Y^T)_{\alpha\beta})$. Thus according to Lemma 1, we conclude that $r_\beta = s_\beta = 0$. Hence $r = s = 0$, which is a contradiction. This establishes the proposition.

<u>Remark 1</u>. The matrices X and Y in Proposition 1 are not required to be Z-matrices.

The following corollary follows immediately from Theorem 1.

<u>Corollary 2</u>. Let $M \in \mathbf{C} \cap R^{n \times n}$. If M satisfies either one of the following two conditions, then $M \in P$:

(3) $M \geq N$ for some S-matrix N;

(4) $M \geq 0$ and M has no vanishing rows.

We now identify several classes of matrices belonging to $\mathbf{C} \cap P$.

<u>Corollary 3</u>. Let $M \in R^{n \times n}$. If M satisfies any one of the following conditions then $M \in \mathbf{C} \cap P$:

(5) $M = Y + ab^T$ for some K-matrix Y and nonnegative vectors a and b;

(6) $M = 2A - B$ for some Z-matrices A and B with $B \in K$ and $A \geq B$;

(7) $M = I + \sum_{i=1}^{k} \alpha_i A^i$ where $A \in R^{n \times n}$ is nonnegative and $\rho(A) < 1$, $1 \geq \alpha_i \geq \alpha_{i+1} \geq 0$ for $i = 1,\ldots,k-1$ and $1 < k \leq \infty$. If $k = \infty$, then it is required in addition that $\rho(A) < \bar{\rho}$ where $\bar{\rho}$ is the radius of convergence of the scalar power series $\sum_{i=1}^{\infty} \alpha_i x^i$;

(8) $M = e^A$, $I + \sinh A$, $I + \cosh A$ where $A \in R^{n \times n}$ is nonnegative and $\rho(A) < 1$;

(9) $M \geq 0$, $\rho(M) < \frac{1}{2}$, $2M \leq (I-M)^{-1}$ and M has no vanishing rows.

Proof: A matrix M satisfying any one of these conditions has been shown to be hidden Z. For (5) and (6), see [9]. For (7), (8) and (9), see [25]. A matrix M satisfying (5) or (6) clearly satisfies (3). A matrix M satisfying (7), (8) or (9) clearly satisfies (4). Therefore, by Corollary 2, we have $M \in P$.

It is well-known that a matrix M is in class K if and only if M can be represented as

(iii) $$M = sI - P$$

where $s > \rho(P)$ and $P \geq 0$. In fact, this representation was used originally by Ostrowski in defining K-matrices [30]. The following theorem generalizes this representation to the class $\mathbf{C} \cap P$.

Theorem 2. Let $M \in R^{n \times n}$. The following are equivalent:

(10) $M \in \mathbf{C} \cap P$;

(11) $M = (s_1 I - P_1)(s_2 I - P_2)^{-1}$ for some nonnegative matrices P_1 and P_2, positive scalars s_1 and s_2 which satisfy the condition below

(iv) $$0 \leq (P_1 u, P_2 u) < (s_1 u, s_2 u) \quad \text{for some } u \in R^n.$$

In particular, if $M = (I - P_1)(I - P_2)^{-1}$ where P_1 and P_2 are sub-stochastic matrices then $M \in \mathbf{C} \cap P$.

Proof: (10) $\Rightarrow$ (11). Suppose $M \in \mathbf{C} \cap P$. Let X and Y be Z-matrices satisfying conditions (M1) and (M2). By the proof of Corollary 1, it follows that there exists a nonnegative vector u such that $Xu > 0$ and $Yu > 0$. Therefore, applying the representation (iii) to X and Y, we obtain (11) readily.

(11) $\Rightarrow$ (10). Let $X = s_2 I - P_2$ and $Y = s_1 I - P_1$. Then (iv) implies that both X and Y are K-matrices. In fact we have $Xu > 0$ and $Yu > 0$. Therefore the matrix $M = YX^{-1} \in \mathbf{C}$. Moreover, with $x = Xu$, we have $x > 0$ and $Mx > 0$. Hence $M \in P$.

The last conclusion of the theorem is obvious. This completes the proof of the theorem.

Remark 2. If M has the representation (11), in particular, if condition (iv) is satisfied, then it follows that

(v) $$s_i > \rho(P_i) \quad \text{for} \quad i = 1,2,$$

or equivalently, both $(s_1 I - P_1)$ and $(s_2 I - P_2)$ are K-matrices. Nevertheless, if both P_1 and P_2 and non-vanishing, then condition (v) alone is not sufficient for M to be a P-matrix. This is illustrated in the example below.

Example 1. Let $P_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $P_2 = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$, $s_1 = 8$ and $s_2 = 5$. It can easily be shown that $s_i > \rho(P_i)$ for $i = 1,2$. Nevertheless the matrix

$$M \equiv (s_1 I - P_1)(s_2 I - P_2)^{-1}$$

$$= \begin{pmatrix} 26 & 19 \\ -8 & -5 \end{pmatrix}$$

which must necessarily be hidden Z, is obviously not a P-matrix.

For $A \in R^{n \times n}$, we define its companion matrix $\mathcal{M}(A) = (m_{ij}) \in R^{n \times n}$ by

$$\mathcal{M}_{ii} = |a_{ii}|, \ \mathcal{M}_{ij} = -|a_{ij}| \quad i \neq j, \quad 1 \leq i, j \leq n.$$

Clearly, $\mathcal{M}(A) \in Z$. The matrix A is said to be an H-matrix [30] if $\mathcal{M}(A) \in K$. A short survey on H-matrices has been given in [35]. See also [40]. It has been shown [29] that the class of H-matrices includes those matrices that are strictly or irreducibly diagonally dominant. Together with Corollary 3, the following proposition shows that the class of H-matrices is a subclass of $\mathbf{C} \cap P$.

Proposition 2. Let $M \in R^{n \times n}$. Then the following are equivalent:

(12) M is an H-matrix.

(13) M satisfies condition (6).

(14) There exist Z-matrices A, B and C with $A \geq C$, $B \geq C$ and $C \in K$ such that $M = \alpha A + B - C$ for some $\alpha \geq 1$.

Moreover, if M satisfies any one of these conditions, then $M \in \mathbf{C} \cap P$.

Proof: (12) $\Rightarrow$ (13). See [30].

(13) $\Rightarrow$ (14). This is obvious.

(14) $\Rightarrow$ (12). According to [17, Thm. 4.6], it suffices to show that $\mathcal{M}(M) \geq \alpha C$. This follows readily if we write down the entries of $\mathcal{M}(M)$ and apply the conditions on A, B, C and α .

The last conclusion of the proposition is an immediate consequence of Corollary 3. This completes the proof of the theorem.

The next example shows that the class of H-matrices is properly contained in $\mathbf{C} \cap P$.

Example 2. Let $M = \begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}$. Then M is a hidden Z-matrix because

$$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -1 \\ -3 & 0 & 2 \end{pmatrix} .$$

Nevertheless there exist no K-matrices A such that $M \geq A$. Indeed if A were such a matrix, then we would have $A \leq \begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. According to Corollary 2, this would imply that $\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \in K$ which is impossible because $\det\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = 0$. Therefore, in particular, M can not be an H-matrix. Moreover, this matrix M satisfies none of the conditions (5)-(9) identified in Corollary 3.

It is well-known that if M is a K-matrix, then M^{-1} exists and is nonnegative. Therefore M^{-1} can not be a Z-matrix except in the trivial case where M is a positive diagonal matrix. Nevertheless, $M^{-1} \in \mathbf{C} \cap P$. More generally, assertion (15) below shows that the class $\mathbf{C} \cap P$ is invariant under inversion.

Proposition 3. Let $M \in \mathbf{C} \cap P$. Then the following are true:

(15) The inverse of M belongs to $\mathbf{C} \cap P$.

(16) Every principal rearrangement of M belongs to $\mathbf{C} \cap P$.

(17) Every principal submatrix of M belongs to $\mathbf{C} \cap P$.

(18) Every principal pivot transform of M belongs to $\mathbf{C} \cap P$.

(19) Every Schur complement in M belongs to $\mathbf{C} \cap P$.

<u>Proof</u>: (15). This is obvious. Simply interchange the roles of X and Y. In fact, this assertion is a special case of (18).

(16). This is also obvious.

(17). This is contained in the proof of Theorem 1. See (i) and (ii).

(18) and (19). These are immediate consequences of (16), (17) and the lemma below.

<u>Lemma 2</u>. Let $M \in \mathbf{C} \cap P \cap R^{n \times n}$. Let X and Y be Z-matrices satisfying (M1) and (M2). Suppose M is partitioned into $M = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}$ where α and β are two complementary index sets in $\{1,\ldots,n\}$. Let X and Y be partitioned into

$$X = \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} \quad Y = \begin{pmatrix} Y_{\alpha\alpha} & Y_{\alpha\beta} \\ Y_{\beta\alpha} & Y_{\beta\beta} \end{pmatrix}$$

accordingly. Then

$$\text{(vi)} \quad \begin{pmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\beta} \\ M_{\beta\alpha} M_{\alpha\alpha}^{-1} & M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta} \end{pmatrix} \begin{pmatrix} Y_{\alpha\alpha} & Y_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ Y_{\beta\alpha} & Y_{\beta\beta} \end{pmatrix} .$$

<u>Proof</u>: We have

$$(M_{\alpha\alpha} \; M_{\alpha\beta}) \begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = (Y_{\alpha\alpha} \; Y_{\alpha\beta})$$

or equivalently,

$$M_{\alpha\alpha} X_{\alpha\alpha} + M_{\alpha\beta} X_{\beta\alpha} = Y_{\alpha\alpha}$$

$$M_{\alpha\alpha} X_{\alpha\beta} + M_{\alpha\beta} X_{\beta\beta} = Y_{\alpha\beta} .$$

Premultiplying $M_{\alpha\alpha}^{-1}$ throughout these latter two equalities and rearranging terms, we obtain

$$\text{(vii)} \quad (M_{\alpha\alpha}^{-1} \; -M_{\alpha\alpha}^{-1} M_{\alpha\beta}) \begin{pmatrix} Y_{\alpha\alpha} & Y_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = (X_{\alpha\alpha} \; X_{\alpha\beta}) .$$

Similarly, we have

$$(M_{\beta\alpha}\ M_{\beta\beta})\begin{pmatrix} X_{\alpha\alpha} & X_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = (Y_{\beta\alpha}\ Y_{\beta\beta})$$

or equivalently,

$$M_{\beta\alpha} X_{\alpha\alpha} + M_{\beta\beta} X_{\beta\alpha} = Y_{\beta\alpha}$$
$$M_{\beta\alpha} X_{\alpha\beta} + M_{\beta\beta} X_{\beta\beta} = Y_{\beta\beta} \ .$$

Substituting the expression (vii) for $(X_{\alpha\alpha}, X_{\alpha\beta})$ into these latter two equalities and rearranging terms, we obtain

$$(M_{\beta\alpha} M_{\alpha\alpha}^{-1} \quad (M/M_{\alpha\alpha}))\begin{pmatrix} Y_{\alpha\alpha} & Y_{\alpha\beta} \\ X_{\beta\alpha} & X_{\beta\beta} \end{pmatrix} = (Y_{\beta\alpha}\ Y_{\beta\beta}) \ .$$

This completes the proof of the lemma.

We conclude this paper by discussing a few points about nonnegative matrices M whose inverses are K-matrices. Such matrices M certainly belong to $\mathcal{C} \cap P$. The next proposition shows that all principal submatrices and Schur complements of such matrices M have inverses which are also K-matrices.

<u>Proposition 4</u>. Let $M \in R^{n \times n}$ be such that M^{-1} is a K-matrix. Then the following are true:

(20) If $M_{\alpha\alpha}$ is a principal submatrix of M, then $M_{\alpha\alpha}^{-1} \in K$;

(21) If $M_{\alpha\alpha}$ is a principal submatrix of M, then $(M/M_{\alpha\alpha})^{-1} \in K$. In particular, $(M/M_{\alpha\alpha})$ is nonnegative.

<u>Proof</u>: In fact, we have $MX = I$ where $X = M^{-1} \in K$. Conclusion (20) follows from (i) which gives $M_{\alpha\alpha}(X/X_{\beta\beta}) = I$ and from the fact that $(X/X_{\beta\beta}) \in K$. Here β is the complement of α in $\{1,\ldots,n\}$. Similarly, conclusion (21) follows from (vi) which gives $(M/M_{\alpha\alpha})X_{\beta\beta} = I$. This completes the proof of the proposition.

<u>Remark 3</u>. As a matter of fact, the two equalities

$$M_{\alpha\alpha}(X/X_{\beta\beta}) = I \quad \text{and} \quad (M/M_{\alpha\alpha})X_{\beta\beta} = I$$

used in the proof of Proposition 4 are direct consequences of the following explicit formula for the inverse of a matrix in partitioned form (see [4] e.g.): if

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ,$$

then

$$W^{-1} = \begin{pmatrix} (W/A)^{-1} & -A^{-1}B(W/A)^{-1} \\ -D^{-1}C(W/D)^{-1} & (W/D)^{-1} \end{pmatrix} .$$

Remark 4. Markham [26] showed that if $M \in R^{n \times n}$ is such that $M^{-1} \in K$, then $M_{\alpha\alpha}^{-1} \in K$ for every α of order $n - 1$. Conclusion (20) is a generalization as well as a consequence of this result.

The condition that each of the proper principal submatrix of a matrix M has an inverse which is a K-matrix is not sufficient for M^{-1} itself to be a K-matrix even when M is nonnegative and a P-matrix. This is illustrated by the following example:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} .$$

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Let $\mathbf{C}$ denote the class of hidden Z-matrices, i.e., $M \in \mathbf{C}$ (an element of) if and only if there exist Z-matrices X and Y such that the following two conditions are satisfied:

(M1) $MX = Y$; and

(M2) $r^T X + s^T Y > 0$ [Transpose] for some $r, s \geq 0$. (> or =)

Let P denote the class of real square matrices having positive principal minors. The class $\mathbf{C}$ arises recently as a generalization of the class of Z-matrices [9] intersecting with [23] ~~, [24]~~. In this paper, ~~we explore~~, various matrix-theoretic aspects of the class $\mathbf{C} \cap P$ are examined.

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